Compressing a One-dimensional One-atom Gas

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Our gas is just a single atom bouncing classically in a one-dimensional chamber with a moveable piston on one end. We want to know what happens when the piston pushes in towards the atom. We constructed this thought experiment to give a kinetic reasoning for the familiar $PV = nkT$ and hope to recover that equation.

We will denote the speed of the atom by v_a and its initial speed before compression by v_0. The speed of the piston is v_p. The cavity begins at a length l_0 so that the moving piston makes the cavity length $l = l_0 - v_p t$ at any point in time.

As the piston compresses, the atom gains velocity by colliding classically with the piston. In a thermodynamic system, the velocity of the piston would not matter. The more slowly it moves, the more times the atom will collide so that the speed of the atom depends only on the cavity length. If the piston speed is much slower than the atom speed, then the velocity of the atom depends only on the length of the cavity.

What are P, V, and T for a one-dimensional one-atom gas? Our new equivalent to volume is the cavity length, l. The temperature can be found from the equipartition of energy

$$\frac{1}{2}mv_a^2 = \frac{1}{2}k_B T \tag{1}$$

where k_B is Boltzmann's constant and m is the atom's mass. The factor on the right is only one half because the atom moves in one dimension (three dimensions, three halves). Solving the above equation for the temperature gives

$$T = \frac{mv^2}{k_B}. \tag{2}$$

The last variable of concern is the pressure. Pressure is normally the force per unit area exerted on the piston. In a two-dimensional gas, pressure would be a force per unit length. Here, pressure is just the average force the atom exerts on the piston. We can find an average force by averaging the familiar Newton's law

$$F = ma = m\frac{\Delta v}{\Delta t}. \tag{3}$$

During a collision with the piston, the atom velocity changes from $+v_a$ to $-v_a$ for a total change of $\Delta t = 2v_a$. The total time per collision is the time it takes the atom to travel down the chamber and back or $t = 2l/v_a$. Our pressure is then

$$P = F = m\frac{2v_a}{2l/v_a} = \frac{mv_a^2}{l}. \tag{4}$$

Do these variables, P, V, and T, fit into an equation of state?

$$\left(\frac{mv_a^2}{l}\right) \cdot l = 1 \cdot k \cdot \left(\frac{mv_a^2}{k}\right). \tag{5}$$

It is exactly $PV = nkT$ where $n = 1$ for one atom.

We can start from examining the simple kinetic collision and show that, for a slowly moving piston, the velocity is a function of the cavity length. When the atom hits the fixed end, it just bounces, but when it hits the compressing piston, it gains the speed of the piston.

$$v_a' = -v_a + 2v_p \tag{6}$$

At any point in time, the velocity of the atom is equal to its initial velocity plus additions from every piston collision

$$v_a = v_0 + 2 \sum_{\text{piston hits}} v_p. \tag{7}$$

Instead of summing over the number of times the piston hit, we could sum over the total time and factor in the number of hits per time.

$$v_a = v_0 + 2 \sum_{\text{time}} v_p \times \frac{\text{hits}}{\text{time}} \tag{8}$$

The frequency of hits is determined by the atom's speed and path length

$$2l = v_a t \quad \Rightarrow \quad \frac{1}{t} = \frac{v_a}{2l} = \frac{v_a}{2(l_0 - v_p t)} \tag{9}$$

We can put this back in the sum to find

$$v_a = v_0 + \sum_t 2v_p \frac{v_a}{2(l_0 - v_p t)} = v_0 + \int \frac{v_p v_a}{(l_0 - v_p t)} dt. \tag{10}$$

We can't do that integral because $v_a = v_a(t)$, and it is unknown. We can rewrite the equation in terms of length using $v_p dt = dl$

$$v_a = v_0 - \int_{l_0}^{l} \frac{v_a}{l} dl \tag{11}$$

Taking the derivative of both sides gives a tractable differential equation

$$\frac{dv_a}{v_a} = -\frac{dl}{l} \tag{12}$$

whose solution is

$$\frac{v_a}{v_0} = \frac{l_0}{l}. \tag{13}$$

Squaring both sides, we could write it as

$$\left(\frac{mv_a^2}{l}\right) \cdot l^3 = \left(\frac{mv_0^2}{l_0}\right) \cdot l_0^3. \tag{14}$$

Looking above at the definitions for pressure and volume in this system, we see our relation between cavity length and velocity is equivalent to

$$PV^3 = \text{constant}. \tag{15}$$

This is not the standard relation for adiabatic compression but is of similar form.